

THE FINITE BASIS AND FINITE RANK PROPERTIES FOR PSEUDOVARIETIES OF SEMIGROUPS

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ABSTRACT. The finite basis property is often connected with the finite rank property. For locally finite varieties and finitely generated pseudovarieties, the two properties are in fact equivalent. In this paper, we construct an example which shows that they are not equivalent in the context of pseudovarieties of semigroups.

1. INTRODUCTION

Although we are mostly interested in semigroups, many considerations in this preliminary and motivational section apply without change to more general algebraic structures. For this reason, we consider algebras of a fixed signature σ , where we assume that σ is finite and consists of symbols of fully defined operations of finite arity. Recall that a (pseudo)variety is a class of algebras closed under the operators of taking homomorphic images, subalgebras and (finitary) direct products. By theorems of Birkhoff [2] and Reiterman [3], we may define a (pseudo)variety by a set of (pseudo)identities, which is called a *basis*. We say that a (pseudo)variety is finitely based if it admits a finite basis of (pseudo)identities. The reader is referred to [1] for more details and basic definitions on this topic.

In Section 2, we recall the definition of (axiomatic) rank of a (pseudo)variety and investigate the connection of the finite basis and finite rank properties. If a (pseudo)variety is finitely based, then it has finite rank and these two properties are equivalent for locally finite (pseudo)varieties. This is why most proofs that concrete finite algebras are infinitely based, in the sense that they generate infinitely based (pseudo)varieties, establish directly that the generated (pseudo)varieties have infinite rank. A case which has deserved a lot of attention is that of finite semigroups, where the main open question is whether one can effectively decide whether a given finite semigroup is finitely based [5]. Another case of interest is that of varieties generated by a semigroup given by a one-relator presentation, for which again the two properties turn out to be equivalent [4].

The two properties, having finite rank and being finitely based, are not, in general, equivalent. Examples of varieties of semigroups which have finite rank and infinitely based are well known. Yet, no previous examples of pseudovarieties with such a property seem to be available in the literature. Indeed, for a non-locally finite pseudovariety \mathbf{V} , the only technique so far available to prove that \mathbf{V} is infinitely based is to show \mathbf{V} has infinite rank.

2010 *Mathematics Subject Classification.* primary 20M07; secondary 08B05.

Key words and phrases. pseudovarieties, free profinite semigroups, pseudoidentities, finite basis, finite rank.

Thus, to find such examples, new proof techniques need to be devised, and this is the main purpose of this paper.

2. BASES AND RANK

We say that an algebra S is *n-generated* if there is some subset of S with at most n elements that generates S .

Let \mathbf{V} be a pseudovariety of finite algebras. We say that \mathbf{V} *has rank at most n* if, whenever a finite algebra S is such that all its n -generated subalgebras belong to \mathbf{V} , so does S . If there is such an integer $n \geq 0$, then \mathbf{V} is said to have *finite rank* and the minimum value of n for which \mathbf{V} has rank at most n is called the *rank* of \mathbf{V} . If there is no such n , then \mathbf{V} is said to have *infinite rank*.

Similarly, we say that a variety \mathcal{V} of algebras *has rank at most n* if, whenever an algebra S is such that all its n -generated subalgebras belong to \mathcal{V} , so does S . The related notions of finite and infinite rank are defined as for pseudovarieties.

The following folklore result amounts to an easy exercise.

Proposition 2.1. *A (pseudo)variety has rank at most n if and only if it admits a basis of (pseudo)identities in at most n variables.*

Proof. The argument is the same for varieties and pseudovarieties, so we deal only with the former. If the variety \mathcal{V} is defined by identities on n variables, then membership in \mathcal{V} is determined by testing conditions on n , not necessarily distinct, elements, whence it holds for an algebra if and only if all its n -generated subalgebras belong to \mathcal{V} ; thus, \mathcal{V} has rank at most n . Conversely, if the variety \mathcal{V} has rank at most n , then \mathcal{V} is defined by all the identities in at most n variables that hold in \mathcal{V} . \square

The alternative definition of rank suggested by Proposition 2.1 can sometimes be found in the literature as the *axiomatic rank*.

A class \mathcal{C} of algebras is said to be *locally finite* if, for every positive integer n , there is a finite upper bound on the cardinalities of n -generated members of \mathcal{C} . In particular, all finitely generated members of a locally finite class are finite. For varieties, this property is equivalent to local finiteness, while every pseudovariety of finite algebras obviously has this property, but there are plenty of non-locally finite pseudovarieties. Furthermore, it is easy to show that a (pseudo)variety generated by a single (or a finite number of) finite algebra(s) is locally finite.

For a variety \mathcal{V} , we denote by \mathcal{V}^F the pseudovariety consisting of all finite members of \mathcal{V} . On the other hand, for a class \mathcal{C} of algebras, we denote by $\langle \mathcal{C} \rangle$ the variety generated by \mathcal{C} . If \mathbf{V} is a pseudovariety, then $\mathbf{V} \subseteq \langle \mathbf{V} \rangle^F$, but the inclusion may be strict. However, it is easy to show that $\mathbf{V} = \langle \mathbf{V} \rangle^F$ if \mathbf{V} is a locally finite pseudovariety. In particular, if the pseudovariety \mathbf{V} is locally finite, then so is the variety $\langle \mathbf{V} \rangle$.

We say that a (pseudo)variety is *finitely based* if it admits a finite basis of (pseudo)identities. Otherwise, we say that it is *infinitely based*.

Proposition 2.1 says nothing about the cardinality of a basis. In the locally finite case, we can say more.

Proposition 2.2. *If a (pseudo)variety is finitely based, then it has finite rank. The converse holds for locally finite (pseudo)varieties.*

Proof. The first part follows from Proposition 2.1. For the converse, suppose first that the locally finite variety \mathcal{V} has finite rank n . Let F be the \mathcal{V} -free algebra on n free generators. Then F satisfies the same identities as \mathcal{V} and, therefore, F generates the variety \mathcal{V} . Since F is finite, so is its diagram Δ , describing the tables of the basic operations on F , where each element of F is represented by a fixed term in the n free generators. It follows that Δ may be viewed as a finite set of identities such that an n -generated algebra belongs to \mathcal{V} if and only if it satisfies Δ . Hence, Δ is a finite basis of identities for \mathcal{V} .

Suppose next that \mathbf{V} is a locally finite pseudovariety of finite rank n . Let F be an n -generated algebra of \mathbf{V} of maximum cardinality. Then, a finite n -generated algebra A belongs to \mathbf{V} if and only if there is an onto homomorphism $F \rightarrow A$, that is, if and only if A satisfies all the identities in the diagram Δ of F . Since \mathbf{V} has rank n , we deduce that Δ is a finite basis of \mathbf{V} . \square

Note that the argument at the end of the proof of Proposition 2.2 does not show that a locally finite pseudovariety of finite rank is finitely generated. The pseudovariety $\llbracket x^2 = x \rrbracket$ of bands is a well-known counter-example of rank 1. However, since finitely generated pseudovarieties are locally finite, we obtain the following result.

Corollary 2.3. *Let \mathbf{V} be a finitely generated pseudovariety. Then, \mathbf{V} is finitely based if and only if \mathbf{V} has finite rank.* \square

Here are some examples of ranks of pseudovarieties of semigroups. The trivial pseudovariety $\mathbf{1} = \llbracket x = y \rrbracket$ is the only pseudovariety of rank 0. The pseudovariety $\mathbf{A} = \llbracket x^{\omega+1} = x^\omega \rrbracket$ of all finite aperiodic semigroups has rank 1. For each $n \geq 1$, the pseudovariety of semigroups

$$\mathbf{V}_n = \llbracket x_1 \cdots x_n = x_1^2, xy = yx \rrbracket$$

has rank n because the semigroup with zero presented by

$$\langle a_1, \dots, a_n; a_i^2 = 0, a_i a_j = a_j a_i \ (\forall i, j) \rangle$$

does not belong to \mathbf{V}_n but all its $(n-1)$ -generated subsemigroups do.

Throughout the paper, when we write a (pseudo)identity of the form $u = 0$ we mean an abbreviation for $uy = u = yu$, where y is a variable that does not occur in u . The next example is taken from [1, Proposition 4.3.14]. The variety of semigroups

$$\mathcal{V} = [x^2 = 0 = xyxzx = xy_1 \cdots y_k xy_k \cdots y_1 \ (\forall k \geq 2)]$$

is locally finite and infinitely based. The proof consists in showing that no finite subset of the defining basis of identities also defines \mathcal{V} , which yields the result by the completeness theorem for equational logic. Since \mathcal{V} is locally finite, thanks to the first two defining identities, it follows from [1, Lemma 4.3.3] that the equational pseudovariety \mathcal{V}^F does not admit a finite basis of identities. It is infinitely based because, in the presence of the identity $x^2 = 0$, every pseudoidentity is equivalent to an identity. Hence, both \mathcal{V} and \mathcal{V}^F have infinite rank by Proposition 2.2.

In conclusion, this section explains why, so far, all proofs in the literature that a variety is infinitely based appear to be either proofs that the variety satisfies the stronger property of having infinite rank or from a given basis of identities one cannot extract a finite one. Similarly, for a pseudovariety \mathbf{V} , proofs that \mathbf{V} is infinitely based are again proofs that \mathbf{V} has infinite rank (which is an equivalent property in case \mathbf{V} is finitely generated) or a proof that a related variety is infinitely based in case \mathbf{V} is equational and locally finite. Moreover, it appears that in all previously known cases of pseudovarieties \mathbf{V} that are infinitely based, \mathbf{V} has infinite rank.

3. INFINITELY BASED FINITE RANK PSEUDOVARIETIES OF SEMIGROUPS

The purpose of this section is to exhibit an example of a pseudovariety of semigroups of finite rank that is infinitely based. The proof that our example enjoys the latter property depends on a suitable reduction to an equational problem. While this allows us to illustrate techniques that may be used to establish that pseudovarieties are infinitely based, by no means do we claim that every infinitely based pseudovariety may be shown to be so by the present method.

Recall that $\mathbf{N} = \llbracket x^\omega = 0 \rrbracket$ is the pseudovariety of all finite *nil* (or *power nilpotent*) semigroups. Also let \mathbf{N}_n be the variety defined by the identity $x_1 \cdots x_n = 0$, where the x_i are distinct variables.

Our candidate for an infinitely based pseudovariety that has finite rank is the following:

$$(3.1) \quad \mathbf{V} = \mathbf{N} \cap \llbracket xy^n x^n y = yx^n y^n x : n \geq 1 \rrbracket.$$

Note that \mathbf{V} contains all finite commutative nil semigroups and, therefore, in particular, it contains all monogenic aperiodic semigroups.

Clearly, \mathbf{V} has rank at most 2. It is defined by an infinite basis and we claim that it has no finite basis. The interest in such an example stems from the fact that new techniques need to be devised to prove that \mathbf{V} is infinitely based.

We start by observing that the rank of \mathbf{V} is precisely 2. Suppose, to the contrary that \mathbf{V} has rank at most 1. By Proposition 2.1, it follows that \mathbf{V} is defined by some set of pseudoidentities in one variable. This is impossible since $\mathbf{V} \subseteq \mathbf{N} \subsetneq \mathbf{A}$ but every aperiodic monogenic semigroup belongs to \mathbf{V} . Hence, \mathbf{V} has rank 2.

Theorem 3.1. *The pseudovariety \mathbf{V} defined by (3.1) is infinitely based.*

Our strategy for proving Theorem 3.1 is to reduce our problem, on pseudoidentities, to a related equational problem, that is a problem about identities. In the present case, this is relatively easy because the pro- \mathbf{N} completion of a finitely generated free semigroup only adds a zero. A key ingredient in the argument is the consideration of suitable finite semigroups. For each $n \geq 1$, denote by S_n the free semigroup on two free generators in the variety

$$\mathbf{V}_n = \mathbf{N}_{2n+3} \cap \llbracket xy^m x^m y = yx^m y^m x : 1 \leq m < n \rrbracket.$$

Lemma 3.2. *The semigroup S_n belongs to \mathbf{N} and it satisfies the identity $xy^m x^m y = yx^m y^m x$ if and only if $n \neq m$.*

Proof. Because $\mathcal{V}_n \subseteq \mathcal{N}_{2n+3}$, the 2-generated semigroup S_n has at most $2^{2n+4} - 1$ elements. In particular, since \mathcal{N}_{2n+3} consists of nil semigroups, S_n belongs to \mathbf{N} .

By definition of S_n , it satisfies all identities of the form $xy^m x^m y = yx^m y^m x$ with $m \neq n$: for $m < n$, because they are defining identities for the variety \mathcal{V}_n ; for $m > n$, because the identity $x_1 \cdots x_{2n+3} = 0$ entails that both sides of $xy^m x^m y = yx^m y^m x$ are zero. So, it remains to show that S_n fails the identity $xy^n x^n y = yx^n y^n x$. For this purpose, since both sides of the identity are too short to apply the identity $x_1 \cdots x_{2n+3} = 0$ and for all the other defining identities for \mathcal{V}_n both sides have the same length, it suffices to show that the identity $xy^n x^n y = yx^n y^n x$ is not a consequence of the identities $xy^m x^m y = yx^m y^m x$ with $m < n$. This follows from the fact that, for $m < n$, if a word that is obtained from $xy^m x^m y$ by substituting x and y by words in two letters u and v , respectively, is a factor of $xy^n x^n y$, then u and v are both powers of the same letter, x or y , whence the equality $uv^m u^m v = vu^m v^m u$ is trivial. Thus, there is no way to change the word $xy^n x^n y$ by using the identities $xy^m x^m y = yx^m y^m x$ with $m < n$ and the claim follows from the completeness theorem for equational logic. \square

Note that Lemma 3.2 implies that none of the defining identities $xy^m x^m y = yx^m y^m x$ may be dropped from the given basis of \mathbf{V} . In particular, the given basis contains no finite basis of pseudoidentities. This is not sufficient to prove that \mathbf{V} is infinitely based because there is no complete finite system of finitary deduction rules for pseudoidentities, that is, there is no analog for pseudoidentities of the completeness theorem for equational logic [1, Theorem 3.8.8]. Yet, the defining pseudoidentities for \mathbf{V} are sufficiently special to allow a syntactic deductive reasoning.

For a word w and a letter a , we denote by $|w|_a$ the number of occurrences of a in w . We say that an identity $u = v$ is *balanced* if $|u|_x = |v|_x$ for every variable x that appears in the identity. Also, we denote by $c(w)$ the set of letters that occur in w .

To circumvent the absence of a completeness theorem for pseudoidentities, we invoke the general theory developed in [1, Section 3.8]. We start by recalling some notation and terminology. We fix a sequence $(x_n)_{n \geq 1}$ of distinct variables and we denote by $\overline{\Omega}_n \mathbf{N}$ the pro- \mathbf{N} semigroup freely generated by $\{x_1, \dots, x_n\}$. For $m < n$, we view $\overline{\Omega}_m \mathbf{N}$ as being naturally embedded in $\overline{\Omega}_n \mathbf{N}$ via the unique continuous homomorphism that sends each x_i ($i = 1, \dots, m$) to itself. This leads to a topological semigroup $\tilde{\Omega}_\omega \mathbf{N}$ which is the inductive limit of the sequence $\overline{\Omega}_1 \mathbf{N} \rightarrow \overline{\Omega}_2 \mathbf{N} \rightarrow \overline{\Omega}_3 \mathbf{N} \rightarrow \dots$. Note that, since each $\overline{\Omega}_n \mathbf{N}$ is the one-point compactification of the discrete free semigroup $\{x_1, \dots, x_n\}^+$, obtained by adding a zero, $\tilde{\Omega}_\omega \mathbf{N}$ is also obtained from the discrete free semigroup $\{x_1, x_2, \dots\}^+$ by adding a zero.

A subset K of $\tilde{\Omega}_\omega \mathbf{N}$ is clopen if and only if $K \cap \overline{\Omega}_n \mathbf{N}$ is clopen for every $n \geq 1$, that is, $K \cap \overline{\Omega}_n \mathbf{N}$ is a finite subset of $\{x_1, \dots, x_n\}^+$ or it is the complement in $\overline{\Omega}_n \mathbf{N}$ of such a subset. In other words, K is clopen in $\tilde{\Omega}_\omega \mathbf{N}$ if and only if $0 \notin K$ and each intersection $K \cap \overline{\Omega}_n \mathbf{N}$ is finite, or $0 \in K$ and each set $\{x_1, \dots, x_n\}^+ \setminus K$ is finite.

Note that the space $\tilde{\Omega}_\omega \mathbf{N}$ is not compact: for instance, the sequence $(x_n)_n$ has no convergent subsequence. In fact, a sequence of words $(u_n)_n$ converges in $\tilde{\Omega}_\omega \mathbf{N}$ if and only if it is eventually constant, or there is N such that $c(u_n) \subseteq \{x_1, \dots, x_N\}$ for all n and $|u_n| \rightarrow \infty$, in which case the limit is zero. Indeed, for any sequence $(x_n)_n$ in $\tilde{\Omega}_\omega \mathbf{N}$ with only a finite number of terms in each $\tilde{\Omega}_n \mathbf{N}$, its complement is an open set in $\tilde{\Omega}_\omega \mathbf{N}$. As another example, the sequence $(x_1 \cdots x_n)_n$ has no convergent subsequence in $\tilde{\Omega}_\omega \mathbf{N}$ even though the length of its terms tends to infinity.

As in [1, Section 3.8], we consider a subset Λ of $\tilde{\Omega}_\omega \mathbf{N} \times \tilde{\Omega}_\omega \mathbf{N}$, which is viewed as an arbitrary set of pseudoidentities for nil semigroups. We say that Λ is *strongly closed* if Λ is a fully invariant congruence on the semigroup $\tilde{\Omega}_\omega \mathbf{N}$ and the clopen unions of Λ -classes separate the classes of Λ . By [1, Corollary 3.8.4], the strongly closed sets of nil pseudoidentities are precisely the sets of all pseudoidentities that are valid in a pseudovariety of nil semigroups.

To prove Theorem 3.1, we establish the following more general result, which extracts the essential hypotheses for the proof.

Theorem 3.3. *Let Σ be a set of balanced semigroup identities such that the variety $[\Sigma]$ is infinitely based. Then, the pseudovariety $\mathbf{N} \cap \llbracket \Sigma \rrbracket$ is infinitely based.*

Proof. Suppose that $\mathbf{W} = \mathbf{N} \cap \llbracket \Sigma \rrbracket$ admits a finite basis Σ_0 of pseudoidentities. We may as well assume that the pseudoidentity $x^\omega = 0$ belongs to Σ_0 . In the presence of that pseudoidentity, every pseudoidentity that is not its consequence is equivalent to either an identity $u = v$ or to a pseudoidentity of the form $u = 0$, where u is a word. The latter case is excluded for pseudoidentities in Σ_0 because no such pseudoidentity is valid in \mathbf{W} since \mathbf{W} contains all monogenic aperiodic semigroups. In the former case, if the identity $u = v$ is not balanced then, together with the pseudoidentity $x^\omega = 0$, it entails a pseudoidentity of the form $w = 0$, for some word w , which has already been excluded. Thus, we may as well assume that Σ_0 consists of the pseudoidentity $x^\omega = 0$ together with a finite set Σ_1 of balanced identities.

By the above cited results from [1, Section 3.8], an identity $u = v$ is valid in $\mathbf{W} = \mathbf{N} \cap \llbracket \Sigma_1 \rrbracket$, if and only if it belongs to every strongly closed set of nil pseudoidentities containing Σ_1 . We claim that the smallest strongly closed set of nil pseudoidentities containing Σ_1 is the fully invariant congruence Λ on $\{x_1, x_2, \dots\}^+$ generated by Σ_1 together with the trivial pseudoidentity $0 = 0$. Indeed, since the set $\theta = \Lambda \cup \{0 = 0\}$ is certainly contained in every fully invariant congruence on $\tilde{\Omega}_\omega \mathbf{N}$ containing Σ_1 , it suffices to show that θ is strongly closed. For this purpose, we start by noting that the θ -classes are the Λ -classes, which are finite sets since they consist of words of equal length involving the same letters, together with the singleton set $\{0\}$. Since finite subsets of $\{x_1, x_2, \dots\}^+$ are clopen in $\tilde{\Omega}_\omega \mathbf{N}$, to separate a class C different from $\{0\}$ from any other class by a clopen union of classes, it suffices to take C itself.

We thus conclude that an identity $u = v$ is valid in the pseudovariety $\mathbf{N} \cap \llbracket \Sigma_1 \rrbracket$ if and only if it is valid in the variety $[\Sigma_1]$. In particular, we obtain

the equality $[\Sigma_1] = [\Sigma]$, which contradicts the assumption that the variety $[\Sigma]$ is infinitely based. Hence, \mathbb{W} is infinitely based. \square

To deduce Theorem 3.1 from Theorem 3.3, it remains to observe that the variety $[xy^n x^n y = yx^n y^n x : n \geq 1]$ is infinitely based by Lemma 3.2.

Acknowledgments. This work was supported, in part, by CMUP (UID/MAT/00144/2013), which is funded by FCT (Portugal) with national (MCTES) and European structural funds (FEDER), under the partnership agreement PT2020. The work of the second author was also partly supported by the FCT post-doctoral scholarship SFRH/BPD/89812/2012.

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